**Formal Properties of Green’s Functions**

Here are the 6 common green’s functions. For the Heisenberg picture operators, I think we must use the typical U(t,t0) = U0(t,t0)S(t,t0), as I don’t think the mixed phase convention time-development operator is compatible with the adiabatic theorem, which will be used later (see QM/Time-Dependent/Adiabatic Perturbation for discussion of phase convention). The time-ordering operator comes with a -ε, where ε = ±, for transposing the operators if they’re fermions/bosons, i.e., T{A(t)B(t)} = A(t)B(t)θ(t-t´) – εB(t)A(t)θ(t´-t). Note that the t argument is always with the A, and t′ always with the B. And the lesser/advanced GF are the only ones with the sign change.



These GF’s also implicitly carry with them exponential convergence factors that come in with the θ functions: θ(t) → θ(t)e-ηt, θ(-t) → θ(-t)eηt, where η = 0+ is an infinitesimal positive number, eventually taken to zero. Also, I think even the >,< GF’s carry these factors, and should be technically defined as, for instance,

G>(t,t´) = -iθ(t-t´)<AH(t)BH(t´)> - iθ(t´-t)<AH(t)BH(t´)>

→

G>(t,t´) = -iθ(t-t´)e-η(t-t´)<AH(t)BH(t´)> - iθ(t´-t)eη(t-t´)<AH(t)BH(t´)>.

Anyway, since this limit gives us 1 (unless t = ∞, or t´ = -∞), we will just leave it off, to save space, unless necessary. The exponential convergence factors are there, I think, because of the use of the adiabatic theorem:|GS> = Sη(t0,-∞)|GS0>, where the η is there to indicate exponentially slowly turning on the perturbation. This is where eηt factors would show up in calculations. And I believe this convergence factor, initially in S (see Quantum Mechanics/Time-Development/Adiabatic Perturbation), gets transferred to the GF’s and so effectively becomes part of their definition for any calculation involving times in infinite past or future (where there is a e-ηt term in the adiabatic theorem). For finite times, the exponential factors are superfluous, as their limit will always gives us 1.

[the ε = +/- on the commutator means anticommutator/commutator] In order for the Wick expansion, that we’ll use later, to hold, we must have that the background expectation state be an eigenstate of some bilinear H0 (or combination of such) |Ω0>



|Ω0> must also be ‘homogeneous’ – it cannot be a macroscopic quantum state, like a Bose condensate, which would seem to mean that the quantum occupation numbers must all be ~ 1/N. Furthermore this is only valid in the large V limit I think. The excitation vacuum state |0> satisfies these requirements. The operators AH(t), BH(t) can be anything, and may evolve in time according to some Hamiltonian,



The interaction term V(t) may be time-dependent, but typically is not in QFT, as the presumption is that there are no ‘external’ fields coming from ‘out there’ to interact with our fields. We just have fields interacting with other fields.

**Algebraic relationships between the various Green’s functions**

The green’s functions aren’t independent. Some algebraic relations between the green’s functions are as follows…well they’re just as they were in the single particle file:



**Symmetries of GF’s**

Consider a GF evaluated against some state |Ω0>. And presume time-development against a constant H. Constant H would mean energy is conserved. Then for instance our GF should depend only on t-t´, i.e., be time-translationally invariant.



Now if |Ω0> is an eigenstate of H, then:



But note that if we evaluated against states that weren’t eigenstates of H (like ones that were eigenstates of H0, rather), then this would not be time-translationally invariant.

**Green’s functions and the spectral function**

The following discussion will assume that we’re using the expectation basis, |Ω0>, which is an arbitrary state. We assume the operators develop according to a **time-independent** H for the spectral properties to hold. Now let’s look at the Fourier transform of the various green’s functions. These results also follow from the aforementioned file in the QM folder. I’ll just list them:



where,



where Im = |m><m| for instance. Note ω is real throughout. Let’s pause to observe that the poles of G are the *many-body excitations*. But they are modulated by the presence of Ω0. So only excitations that have non-zero overlap with |Ω0> will be present.

**Properties of the Spectral Function**

We can see that the spectral function can be obtained by:



If A(ω) is real, then GA = (GR)\*, and we can say:



Moreover, we saw the Spectral Function was explicitly given by:



So it clearly has poles (infinities) at the excitations of the system (since |m>, |n> are the eigenstates, and so En – Em would be the excitations). But these are modulated by the overlap of the excitation states with |Ω0> and the operators A and B. The reality of A(x) seems to hinge on a lot of things. So if A and B are Hermitian conjugates of each other (or themselves), and if |Ω0> is an eigenstate of H, then A(x) should be real since:



And then its complex conjugate would be:



So there.

**Kramers - Kronig relations obeyed by GF’s and A(ω)**

Repeating the arguments from the single particle file, we have, *presuming* A(ω) is real:



where the Hilbert transform is:



and for the spectral function, we have:



and concommitantly:



**Expectations from GF’s**

One of the utilities of a GF is how we can get expectations from it. Consider a 1D continuous non-harmonic lattice Hamiltonian, in an external time-dependent field:



Suppose we are in some initial state |Ω0>. We let it evolve according to H, and then want to calculate the average spring potential energy as a function of time thereafter. Then we’d calculate:



We can define the causal GF:



And then we’d have:



We could get KE too, using the canonical relation π(x,t) = ∂L/∂(x,t) = μ(x,t) [have to form the Lagrangian to see this of course]. Then,



**Expectations from the spectral function**

From the definition of G<, we have:



We also have,



This property actually follows from the definition of A in terms of GR and GA. So it is completely general. Let’s consider calculating <AB>. We can clearly find the appropriate linear combination of the two above.



**Differential equations obeyed by GF’s**

Let’s consider an interacting phonon system (elastic solid):



where V is some function of the coordinates and time. And let’s consider the retarded GF:



We can take the derivative w/r to time (see QM/TimeDependent file or QM/MultipleParticles/Distinct Particles for similar calculations), and take advantage of the fact that the operator φH will obey classical equations of motion, to arrive at:



The commutator between the two φ’s is zero. And so we’re left with derivative of φ. So let’s work that out,



which is what we expect. So now we have:



Now let’s take the derivative again,



The first term can be evaluated most easily by setting t´ = t (because δ makes it so) factoring out the time-evolution of the operators, evaluating the commuator, and then applying the time-evolution operator again. Of course since the commutator is a constant, this cancels out. And we get:



and now we need to evaluate the derivative of π…



where in the last line we IBP. And then,



Filling this into our GF equation,



and so we have:



where the perturbing force is:



I would imagine the causal, advanced, etc., guys obey similar equations, but note that FC,A,<,>, etc., would also have corresponding changes in form. This whole analysis would’ve been easier if we had just found the equation of motion of φ(x,t) directly from the action itself. But whatever.